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The GHS Inequality for a Large External Field

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We consider general even ferromagnetic systems with pair interactions in a nonnegative external magnetic field h. Classes of single-site measures ρ are found such that the GHS inequality is valid for all $h \geqslant \tilde{h}$, where $\tilde{h} \geqslant 0$ is a number depending on ρ but independent of the size of the system. These measures include both absolutely continuous and discrete measures. For $\rho \doteq a\delta_0 + \{(1-a)/2\} \cdot (\delta_1 + \delta_{-1})$, some $a \in [0, 1)$, $\tilde{h}$ is determined exactly.

KEY WORDS: GHS inequality; general even ferromagnetic systems; correlation inequalities.

1. INTRODUCTION

The Griffiths–Hurst–Sherman (GHS) inequality is a useful tool in the study of lattice spin systems with ferromagnetic pair interactions. For example, when valid, it implies that the average magnetization per site is a concave function of $h \geqslant 0$, where h denotes the external magnetic field. It also implies the absence of spontaneous magnetization except possibly at $h = 0$.[(9)] However, the validity of the GHS inequality for a particular system depends upon that system's single spin measures. For example, it holds for spin-1/2 systems[(5)]—i.e., for systems with single spin measures the Bernoulli measure $\frac{1}{2}(\delta_1 + \delta_{-1})$—but not for systems with single spin

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measures

$$\rho_a \doteq a\delta_0 + \frac{1-a}{2}(\delta_1 + \delta_{-1}) \tag{1.1}$$

with $a \in (2/3, 1)$ [Ref. 6, p. 153; Ref. 3, Theorem 1.2(b)]. In this respect, it differs from the related Griffiths–Kelly–Sherman inequalities (GKS I, II), which hold for systems with arbitrary even single spin measures (Ref. 10, §VIII.3).

Previous work has determined a large class of absolutely continuous measures for which the GHS inequality is valid [Ref. 3, Theorem 1.2(c)–(d)]. This class contains all absolutely continuous measures with densities const $\times \exp(-V)$, where V is an even C^1 function on $\mathbb{R}$, unbounded above at infinity, with dV/dx convex on $[0, \infty)$. The situation for discrete measures is different. Aside from the Bernoulli measure, the measures $\{\rho_a\}$ in (1.1) with $a \in [0, 2/3]$, and the spin-$n/2$ measures ($n \in \{2, 3, \ldots\}$) in Ref. 4, little is known about other discrete, finitely supported measures for which the GHS inequality holds.

The present paper studies a related problem. Consider, for simplicity, a system of N sites all of whose single spin measures coincide with a fixed even measure ρ. We determine large classes of both absolutely continuous and discrete ρ's such that the GHS inequality is valid, not necessarily for all $h \geqslant 0$, but for all h sufficiently large; i.e., $h \geqslant \tilde{h}$, for some $\tilde{h} \geqslant 0$. The number $\tilde{h}$ may depend upon ρ, but it is independent of N. An implication of our work is that in the thermodynamic limit such a system cannot exhibit spontaneous magnetization (i.e., its magnetization is a continuous function of h) for $|h| > \tilde{h}$.

We remark on work of Dunlop which is similar in spirit to our own. In Ref. 2, single spin measures are found such that the pressure is analytic, not necessarily in the region $\{h \mid h \in \mathbb{C}, \mathrm{Re}\, h \neq 0\}$ (as in Ref. 7), but in the region $\{h \mid h \in \mathbb{C}, |\mathrm{Im}\, h| < \mathrm{Re}\, h\}$ of the complex external field h. His approach, like ours, is based on correlation inequalities. We also note that there are unpublished results of D. Ruelle concerning single spin measures for which the pressure is analytic in the region $\{h \mid h \in \mathbb{C}, |\mathrm{Re}\, h| > \tilde{h}\}$ for some $\tilde{h} \geqslant 0$.

Section 2 of this paper states our main results. Section 3 gives additional facts needed for their proofs. The proofs of the main results are given in Section 4.

2. MAIN RESULTS

We consider general even ferromagnetic systems with pair interactions in a nonnegative external magnetic field. Such a system is defined by a finite family of real-valued random variables $\{X_i; i = 1, \ldots, N\}$ with joint probability distribution

$$\tau_{h_1,\ldots,h_N}(dx_1,\ldots,dx_N) \doteq \frac{1}{Z(h_1,\ldots,h_N)}\exp\left(\sum_{i,j=1}^{N} J_{ij}x_ix_j + \sum_{i=1}^{N} h_ix_i\right) \times \prod_{i=1}^{N} \rho_i(dx_i) \tag{2.1}$$

where

$$Z(h_1,\ldots,h_N) \doteq \int_{R^N} \exp\left(\sum_{i,j=1}^{N} J_{ij}x_ix_j + \sum_{i=1}^{N} h_ix_i\right)\prod_{i=1}^{N} \rho_i(dx_i) \tag{2.2}$$

Unless otherwise noted, we assume that $J_{ij} \geqslant 0$, $h_i \geqslant 0$ for all $i, j \in \{1,\ldots,N\}$. The single-spin measures $\{\rho_i; i = 1,\ldots,N\}$ are assumed to belong to $\mathfrak{S}$, which is the class of even probability measures $\rho \neq \delta_0$ satisfying $\int \exp(bx^2)\rho(dx) < \infty$ for all $b > 0$.

We say that $\rho_1,\ldots,\rho_N \in \mathfrak{S}$ satisfy the GHS inequality for a large external field if there exist nonnegative numbers $\{\tilde{h}_i; i = 1,\ldots,N\}$ such that for all $j, k, l \in \{1,\ldots,N\}$ and all nonnegative $\{J_{ij}; i, j = 1,\ldots,N\}$

$$\frac{\partial^3}{\partial h_j \partial h_k \partial h_l} \ln Z(h_1,\ldots,h_N) \leqslant 0 \qquad \text{whenever} \quad h_i \geqslant \tilde{h}_i,\, i \in \{1,\ldots,N\} \tag{2.3}$$

We say that a measure $\rho \in \mathfrak{S}$ satisfies the GHS inequality for large external field if there exists $\tilde{h} \geqslant 0$ depending on ρ such that for all $N \in \{1,2,\ldots\}$, all $j, k, l \in \{1,\ldots,N\}$, and all nonnegative $\{J_{ij}; i, j = 1,\ldots,N\}$

$$\frac{\partial^3}{\partial h_j \partial h_k \partial h_l} \ln Z_\rho(h_1,\ldots,h_N) \leqslant 0 \qquad \text{whenever} \quad h_i \geqslant \tilde{h},\, i \in \{1,\ldots,N\} \tag{2.4}$$

In (2.4), $Z_\rho \doteq Z$ in (2.2) with $\rho_1 = \cdots = \rho_N \doteq \rho$.

In Section 3, we define classes of measures $\{\mathcal{G}(\tilde{h}); \tilde{h} \geqslant 0\}$, where each $\mathcal{G}(\tilde{h})$ is defined in terms of an infinite set of correlation inequalities. Our first theorem shows the connection between these classes and the inequalities (2.3) and (2.4).

Theorem 1. Let $\{\tilde{h}_i; i = 1,\ldots,N\}$ be nonnegative numbers and $\rho_1,\ldots,\rho_N$ measures in $\mathfrak{S}$ such that $\rho_i \in \mathcal{G}(\tilde{h}_i)$, $i \in \{1,\ldots,N\}$. Then (2.3) holds. In particular, if $\rho \in \mathfrak{S}$ belongs to $\mathcal{G}(\tilde{h})$ for some $\tilde{h} \geqslant 0$, then (2.4) holds.

The next two theorems, Theorems 2 and 3, give a large class of measures belonging to $\mathcal{G}(\tilde{h})$ for some $\tilde{h} \geqslant 0$. Theorem 2 treats the absolutely continuous case and Theorem 3 the discrete case.

Theorem 2. Suppose that $\rho(dx)$ has the form const $\times \exp[-V(x)]dx$, where V is an even C^2 function on $\mathbb{R}$ such that dV/dx is convex on $[a, \infty)$ for some $a > 0$ and $d^2V/dx^2 \to \infty$ as $x \to \infty$. Then there exists $\tilde{h} \geqslant 0$ such that $\rho \in \mathcal{G}(\tilde{h})$.

Example. If V is an even polynomial of degree $d \geqslant 4$ with positive leading coefficient, then V satisfies the hypotheses of Theorem 2.

For the discrete case, let $\rho \in \mathcal{E}$ be supported on finitely many points. Thus, ρ has the form

$$\rho \doteq \sum_{i=1}^{r} c_i \delta_{m_i} \in \mathcal{E} \tag{2.5}$$

where $r \in \{2, 3, \ldots\}$, $m_1 < \cdots < m_r$, $c_i > 0$, and $\sum_{i=1}^{r} c_i = 1$. Theorem 3 below covers such ρ.

Theorem 3. Let ρ be as in (2.5).

(a) If $r = 2$, then $\rho \in \mathcal{G}(0)$.

(b) If $r \in \{3, 4, 5\}$, then $\rho \in \mathcal{G}(\tilde{h})$ for some $\tilde{h} \geqslant 0$.

(c) If $r \in \{6, 7, \ldots\}$, then suppose the points m_i are equally spaced; i.e.,

$$m_{j+2} - m_{j+1} = m_{j+1} - m_j \qquad \text{for} \quad j = 1, 2, \ldots, r-2 \tag{2.6}$$

If (2.6) holds, then $\rho \in \mathcal{G}(\tilde{h})$ for some $\tilde{h} \geqslant 0$.

(d) There is a $\bar{\rho}$ as in (2.5) with $r = 6$ which is not in $\mathcal{G}(\tilde{h})$ for any $\tilde{h} \geqslant 0$.

Remark. It can also be shown that for $r \in \{6, 7, \ldots\}$, a sufficient condition to ensure that $\rho \in \mathcal{G}(\tilde{h})$ for some $\tilde{h} > 0$ is that

$$m_{j+2} > m_{j+1} + m_j \quad \text{for all} \quad r - 2 \geqslant j \geqslant [(r+1)/2] + 1$$

where the notation $[-]$ denotes the greatest integer function. See Ref. 8, Chap. 4 for the details.

The value of Theorems 2 and 3 would be enhanced if one could determine, or even estimate, the number $\tilde{h}$. While in general this is difficult, for the measures $\{\rho_a; a \in (2/3, 1)\}$ in (1.1) ($r = 3$ in Theorem 3), we have the following result.

Theorem 4. Suppose that $\rho_a = a\delta_0 + [(1-a)/2](\delta_1 + \delta_{-1})$ for some $a \in (2/3, 1)$. Then $\rho_a \in \mathcal{G}(\tilde{h})$, where $\tilde{h} = \tilde{h}(a)$ is given by

$$\cosh[\tilde{h}(a)] = 1 + \frac{3a - 2}{a(1-a)} \tag{2.7}$$

Remarks. (1) If $a \in [0, 2/3]$, then $\rho \in \mathcal{G}(0)$ [see Ref. 3, Theorem 1.2(b)].

(2) The $\tilde{h}$ of (2.7) is the best possible since ρ_a satisfies the single site GHS inequality only for $h \geqslant \tilde{h}(a)$; i.e., if $a \in (2/3, 1)$, then

$$\left\{ h > 0 \,|\, (d^3/dh^3)\ln \int_R \exp(hx)\rho_a(dx) < 0 \right\} = (\tilde{h}(a), \infty) \qquad (2.8)$$

with $\tilde{h}(a)$ given by (2.7).

(3) For general ρ covered by Theorem 2 or Theorem 3, one can derive an upper bound on $\tilde{h}$ in terms of the integer $\tilde{n}$ appearing in Lemma 1 below and a finite number (depending on $\tilde{n}$) of the moments of ρ. The proof is based upon an estimate [Ref. 1, Chap. 3] of the largest positive roots of the polynomials $\{P(\mathbf{n}; h)\}$ in (3.7) below for finitely many $\mathbf{n}$. See [Ref. 8, Chap. 2] for details. However, in general the estimation of $\tilde{n}$ is a formidable task.

3. THE CLASSES $\{\mathcal{G}(\tilde{h})\}$

We first define the classes $\{\mathcal{G}(\tilde{h})\}$, then state a lemma, Lemma 1, which will be the main technical tool for proving Theorems 2 and 3.

Let B be the orthogonal matrix

$$B \doteq \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$

Given $\rho \in \mathcal{E}$, let $\mathbf{W} \doteq (W_1, W_2, W_3, W_4)$ be a random vector with independent components, each distributed by ρ. Let $\mathbb{Z}_+^4$ denote the set of all $\mathbf{n} \doteq (n_1, n_2, n_3, n_4)$, where each n_α is a nonnegative integer, and define

$$(B\mathbf{W})^{\mathbf{n}} \doteq \prod_{\alpha=1}^{4} \{(B\mathbf{W})_\alpha\}^{n_\alpha}$$

where $(B\mathbf{W})_\alpha$ is the αth component of the vector $B\mathbf{W}$. Denote by $E_\rho\{\cdot\}$ expectation with respect to the product measure $\prod_{\alpha=1}^{4}\rho(dw_\alpha)$.

Definition 1. Given $\tilde{h} \geqslant 0$, define

$$\mathcal{G}(\tilde{h}) \doteq \left\{ \rho \,|\, \rho \in \mathcal{E}, E_\rho\left\{(B\mathbf{W})^{\mathbf{n}} \exp\left[2\tilde{h}(B\mathbf{W})_1\right]\right\} \geqslant 0 \quad \text{for all} \quad \mathbf{n} \in \mathbb{Z}_+^4 \right\}$$

The class $\mathcal{G}(0)$ coincides with the class $\mathcal{G}_-$ described in Ref. 3. We note several properties of the $\{\mathcal{G}(\tilde{h})\}$:

(i) If $0 \leqslant \tilde{h}_1 < \tilde{h}_2$, then $\mathcal{G}(\tilde{h}_1) \subseteq \mathcal{G}(\tilde{h}_2)$.

(ii) If $\rho_1, \rho_2 \in \mathcal{G}(\tilde{h})$, then $\rho_1 * \rho_2 \in \mathcal{G}(\tilde{h})$.

(iii) If $\{\rho_i; i = 1, 2, \ldots\}$ is a sequence of measures in $\mathcal{G}(\tilde{h})$, $\rho_i \Rightarrow \rho \in \mathcal{E}$ and $\sup_i \int |x|^n \exp[\tilde{h}x]\rho_i(dx) < \infty$ for each $n \in \{0, 1, 2, \ldots\}$, then $\rho \in \mathcal{G}(\tilde{h})$.

Here, the symbols $*$ and $\Rightarrow$ denote, respectively, convolution of measures and weak convergence of measures. The proofs of these properties are elementary.

In the next lemma and its proof, $\mathbf{n}$ odd means that each n_α, $\alpha \in \{1, \ldots, 4\}$, is odd; $\mathbf{n}$ even is defined similarly.

Lemma 1. Given $\rho \in \mathcal{E}$, suppose that there exists a nonnegative integer $\tilde{n}$ such that

$$E_\rho\{(B\mathbf{W})^{\mathbf{n}}\} > 0 \qquad \text{for all odd } \mathbf{n} \text{ satisfying } n_1 \geqslant \tilde{n} \tag{3.1}$$

Then there exists $\tilde{h} \geqslant 0$ such that $\rho \in \mathcal{G}(\tilde{h})$.

Proof. Since the condition (3.1) is only on odd $\mathbf{n}$, we may assume, without loss of generality, that $\tilde{n}$ is odd. The evenness of ρ implies [Ref. 3, Theorem 2.6(c)]

$$E_\rho\{(B\mathbf{W})^{\mathbf{n}}\}\begin{cases} > 0 & \text{if } \mathbf{n} \text{ is even} \\ = 0 & \text{if } \mathbf{n} \text{ is neither even nor odd} \end{cases} \tag{3.2}$$

and that if (3.1) holds, then

$$E_\rho\{(B\mathbf{W})^{\mathbf{n}}\} > 0 \quad \text{for all odd} \quad \mathbf{n} \in \mathbb{Z}_+^4 \quad \text{satisfying} \quad \max\{n_1, n_2, n_3, n_4\} \geqslant \tilde{n} \tag{3.3}$$

We use these facts below.

For $\mathbf{n} \in \mathbb{Z}_+^4$ and $h \geqslant 0$, we define

$$f(\mathbf{n}; h) \doteq E_\rho\{(B\mathbf{W})^{\mathbf{n}} \exp\{2h(B\mathbf{W})_1\}\} \tag{3.4}$$

Since $\rho \in \mathcal{E}$, $f(\mathbf{n}; \cdot)$ is real analytic. We define the finite set of multi-indices

$$\mathfrak{N} \doteq \{\mathbf{n} \,|\, \mathbf{n} \in \mathbb{Z}_+^4;\, n_2, n_3, n_4 \quad \text{all odd}, \quad \max\{n_1, n_2, n_3, n_4\} < \tilde{n}\} \tag{3.5}$$

We prove that for all $\mathbf{n} \notin \mathfrak{N}$, $f(\mathbf{n}; h) \geqslant 0$ for all $h \geqslant 0$ and that there exists $\tilde{h} \geqslant 0$ such that for all $\mathbf{n} \in \mathfrak{N}$, $f(\mathbf{n}; h) \geqslant 0$ for all $h \geqslant \tilde{h}$. This will prove the lemma.

For fixed $\mathbf{n} \in \mathbb{Z}_+^4$, we expand $f(\mathbf{n}; h)$ in a Taylor series about $h = 0$, noting

$$f^{(k)}(\mathbf{n}; 0) = E_\rho\{(B\mathbf{W})^{(n_1+k, n_2, n_3, n_4)}\}2^k \tag{3.6}$$

for $k \in \{0, 1, 2, \ldots\}$. First assume $\mathbf{n} \notin \mathfrak{N}$. Then either n_2, n_3, n_4 are not all odd or n_2, n_3, n_4 are all odd and $\max\{n_1, n_2, n_3, n_4\} \geqslant \tilde{n}$. In either case, by (3.2), (3.3), $f^{(k)}(\mathbf{n}; 0) \geqslant 0$ for all $k \in \{0, 1, \ldots\}$. Thus, if $\mathbf{n} \notin \mathfrak{N}$, $f(\mathbf{n}; h) \geqslant 0$

for all $h \geqslant 0$. Now assume $\mathbf{n} \in \mathfrak{N}$. We write

$$f(\mathbf{n};h) = \sum_{k=0}^{\tilde{n}-n_1} \frac{f^{(k)}(\mathbf{n};0)}{k!} h^k + \sum_{k=\tilde{n}-n_1+1}^{\infty} \frac{f^{(k)}(\mathbf{n};0)}{k!} h^k \equiv P(\mathbf{n};h) + R(\mathbf{n};h) \tag{3.7}$$

By (3.3), $R(\mathbf{n};h) \geqslant 0$ for all $h \geqslant 0$ and so

$$f(\mathbf{n};h) \geqslant P(\mathbf{n};h) \qquad \text{for} \quad h \geqslant 0 \tag{3.8}$$

By (3.6) the leading coefficient of $P(\mathbf{n};h)$ is proportional to $E_\rho\{(B\mathbf{W})^{(\tilde{n},n_2,n_3,n_4)}\}$, which is strictly positive by (3.1). Hence for each $\mathbf{n} \in \mathfrak{N}$, there exists a number $h'(\mathbf{n}) \geqslant 0$ such that $P(\mathbf{n};h) \geqslant 0$ for $h \geqslant h'(\mathbf{n})$. Defining

$$\tilde{h} \doteq \max\{h'(\mathbf{n}) \mid \mathbf{n} \in \mathfrak{N}\} \tag{3.9}$$

we see by (3.7) that $f(\mathbf{n};h) \geqslant 0$ for $h \geqslant \tilde{h}$. ■

4. PROOFS OF THEOREMS 2, 3, AND 4

Proof of Theorem 1. This theorem is proved like Theorem 1.1 in Ref. 3, which is its analog for $\tilde{h} = 0$. Alternatively, one can easily adapt the proof of the GHS inequality given in Ref. 11. ■

Proof of Theorem 2. Without loss of generality, we may assume that V' is strictly convex on $[a,\infty)$. For if V is convex, then for $\epsilon > 0$, $V_\epsilon(x) \doteq V(x) + \epsilon x^4$ satisfies the hypotheses of the theorem and is strictly convex on $[a,\infty)$. Our proof of Theorem 2 will show that each $\rho_\epsilon(dx) \doteq \text{const} \times \exp[-V_\epsilon(x)]dx \in \mathcal{G}(\tilde{h})$ for some $\tilde{h} \geqslant 0$, and one can prove that $\tilde{h}$ can be picked independent of ϵ for ϵ sufficiently small. By property (iii) of the classes $\{\mathcal{G}(\tilde{h})\}$, we conclude $\rho_0 \in \mathcal{G}(\tilde{h})$.

Define the orthogonal matrix

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Given $\rho \in \mathcal{S}$ and an invertible 4×4 matrix T, we define the measure $\boldsymbol{\rho}_T$ on $\mathbb{R}^4$ by $\boldsymbol{\rho}_T(F) = \boldsymbol{\rho}(T^{-1}F)$, where F is a Borel subset of $\mathbb{R}^4$ and $\boldsymbol{\rho}$ is the product measure $\boldsymbol{\rho}(d\mathbf{w}) \doteq \prod_{\alpha=1}^4 \rho(dw_\alpha)$. Define the signed measure $\boldsymbol{\sigma}$ on $\mathbb{R}^4$ by $\boldsymbol{\sigma} \doteq \boldsymbol{\rho}_B - \boldsymbol{\rho}_A$ and let $\boldsymbol{\sigma} = \boldsymbol{\sigma}_+ - \boldsymbol{\sigma}_-$ be its Jordan decomposition. Finally, define

$$S_+ \doteq \{\text{support of } \boldsymbol{\sigma}_+\} \cap \mathbb{R}^4_+ \quad \text{and} \quad S_- \doteq \{\text{support of } \boldsymbol{\sigma}_-\} \cap \mathbb{R}^4_+$$

where $\mathbb{R}^4_+$ denotes the positive orthant of $\mathbb{R}^4$.

We need two lemmas. The first of these, Lemma 2, is Proposition 4.1 in Ref. 3. The proof of the second, Lemma 3, will be given below, after we complete the proof of Theorem 2. Given $\mathbf{s}, \mathbf{t} \in \mathbb{R}^4$, we write $\mathbf{t} > \mathbf{s}$ if $t_\alpha > s_\alpha$ for each $\alpha \in \{1, 2, 3, 4\}$.

Lemma 2. For $\rho \in \mathfrak{G}$ and all odd $\mathbf{n} \in \mathbb{Z}^4_+$

$$E_\rho\{(B\mathbf{W})^{\mathbf{n}}\} = 8\int_{\mathbb{R}^4_+} \mathbf{w}^{\mathbf{n}} \boldsymbol{\sigma}(d\mathbf{w}) = 8\int_{\mathbb{R}^4_+} \mathbf{w}^{\mathbf{n}} \boldsymbol{\sigma}_+(d\mathbf{w}) - 8\int_{\mathbb{R}^4_+} \mathbf{w}^{\mathbf{n}} \boldsymbol{\sigma}_-(d\mathbf{w})$$

Lemma 3. Under the hypotheses of Theorem 2, S_- is bounded, and given any real number R there exists $\mathbf{t} \in S_+$ with $\mathbf{t} > (R, R, R, R)$.

We now show that there exists an integer $\tilde{n}$ such that $E_\rho\{(B\mathbf{W})^{\mathbf{n}}\} > 0$ for all odd $\mathbf{n}$ satisfying $n_1 \geqslant \tilde{n}$. Lemma 1 then completes the proof of Theorem 2. We define

$$\mu \doteq \sup\{\max\{s_1, s_2, s_3, s_4\} \mid \mathbf{s} \in S_-\}$$

(If $S_- = \varnothing$, then set $\mu \doteq 0$.) By Lemma 3 there exists $\boldsymbol{\xi} \in S_+$ such that $\boldsymbol{\xi} > (\mu, \mu, \mu, \mu)$ and $\sigma_+\{\mathbf{w} \mid \mathbf{w} \geqslant \boldsymbol{\xi}\} > 0$. By Lemma 2,

$$\begin{aligned} \tfrac{1}{8} E_\rho\{(B\mathbf{W})^{\mathbf{n}}\} &\geqslant \boldsymbol{\xi}^{\mathbf{n}} \sigma_+\{\mathbf{w} \mid \mathbf{w} > \boldsymbol{\xi}\} - \mu^{|\mathbf{n}|} \sigma_-\{S_-\} \\ &\geqslant \bar{\xi}^{|\mathbf{n}|} \sigma_+\{\mathbf{w} \mid \mathbf{w} > \boldsymbol{\xi}\} - \mu^{|\mathbf{n}|} \sigma_-\{S_-\} \end{aligned} \tag{4.1}$$

where $\bar{\xi} \doteq \min\{\xi_1, \xi_2, \xi_3, \xi_4\}$ and $|\mathbf{n}| \doteq n_1 + n_2 + n_3 + n_4$. If $\mu = 0$, then $E_\rho\{(B\mathbf{W})^{\mathbf{n}}\} > 0$ for all odd $\mathbf{n}$. If $\mu > 0$, then the right-hand side of (4.1) is strictly positive if

$$\left(\frac{\bar{\xi}}{\mu}\right)^{|\mathbf{n}|} > \frac{\sigma_-\{S_-\}}{\sigma_+\{\mathbf{w} \mid \mathbf{w} > \boldsymbol{\xi}\}} \tag{4.2}$$

This can be achieved for $|\mathbf{n}|$ sufficiently large since $\bar{\xi} > \mu$. Defining $\tilde{n}$ to be the smallest integer $|\mathbf{n}|$ satisfying (4.2), we have verified the hypothesis of Lemma 1. ■

Proof of Lemma 3. By definition of $\boldsymbol{\sigma}$, we have

$$\boldsymbol{\sigma}(d\mathbf{w}) = \left\{\exp\left[-\sum_{\alpha=1}^{4} V\big((B^{-1}\mathbf{w})_\alpha\big)\right] - \exp\left[-\sum_{\alpha=1}^{4} V\big((A^{-1}\mathbf{w})_\alpha\big)\right]\right\} \prod_{\alpha=1}^{4} dw_\alpha$$

Define the function $\tilde{V}$ on $\mathbb{R}^4$ by

$$\tilde{V}(\mathbf{w}) \doteq \sum_{\alpha=1}^{4} \left[V\big((A^{-1}\mathbf{w})_\alpha\big) - V\big((B^{-1}\mathbf{w})_\alpha\big)\right]$$

It is not hard to see that $S_- = \{\mathbf{w} \mid \mathbf{w} \in \mathbb{R}_+^4, \tilde{V}(\mathbf{w}) \leqslant 0\}$. Hence Lemma 3 is proved once we show that there exists $M > 0$ such that if $\mathbf{w} \in \mathbb{R}_+^4$ and $\max\{w_1, w_2, w_3, w_4\} \geqslant M$, then $\tilde{V}(\mathbf{w}) > 0$.

Our assumptions on V are that V is even, V' is strictly convex on the interval $[a, \infty)$, and $V''(x) \to \infty$ as $x \to \infty$. Hence there exists a number $M > a$ such that

$$V''(x) > \sup\{V''(y) \,\big|\, |y| < a\} \qquad \text{for all } |x| \geqslant M$$

We see that $V''(x) > V''(y)$ whenever $|x| \geqslant M$ and $|y| < |x|$. For $w_1, w_2 > 0$, define

$$f(w_1, w_2) \doteq V''(w_1 + w_2) - V''(w_1 - w_2)$$

Then $f(w_1, w_2) > 0$ whenever $w_1 + w_2 \geqslant M$ and thus whenever $\max\{w_1, w_2\} \geqslant M$. A short calculation shows that for $\mathbf{w} > (0,0,0,0)$,

$$\tilde{V}(\mathbf{w}) = \int_{w_1 - w_4}^{w_1 + w_4} dx \int_{x - w_3}^{x + w_3} f(y, w_2)\, dy$$

Since $\tilde{V}$ is unchanged by permutations of the w_α's, we may assume that $w_1 \geqslant w_4$ (so that $x \geqslant 0$ above) and that $w_2 = \max(w_1, w_2, w_3, w_4)$. Since V'' is even, $f(y, w_2)$ is an odd function of y. Thus

$$\tilde{V}(\mathbf{w}) = \int_{w_1 - w_4}^{w_1 + w_4} dx \int_{|x - w_3|}^{x + w_3} f(y, w_2)\, dy$$

which is strictly positive if $w_2 = \max\{w_1, w_2, w_3, w_4\} \geqslant M$. ∎

The proof of Theorem 3 will be based on two lemmas, Lemmas 4 and 5. Given $\rho \in \mathcal{S}$, we define subsets $\mathfrak{M}_+$ and $\mathfrak{M}_-$ of $\mathbb{R}^4$ by

$$\mathfrak{M}_\pm \doteq \left\{ \mathbf{w} = (w_1, w_2, w_3, w_4) \mid \text{each } w_i \in \text{support}(\rho), \right.$$
$$\left. (B\mathbf{w})_1 > 0,\ \pm \prod_{\alpha=1}^{4} (B\mathbf{w})_\alpha > 0 \right\}$$

Lemma 4. Let $\rho \in \mathcal{S}$ be as in (2.5). Suppose that for each $\mathbf{w} \in \mathfrak{M}_-$ there exist $\mathbf{v} = \mathbf{v}(\mathbf{w})$ and $\mathbf{v}' = \mathbf{v}'(\mathbf{w})$, (not necessarily distinct) vectors in $\mathfrak{M}_+$, such that for some $k = k(\mathbf{w}) \in \{1,2,3,4\}$,

$$|(B\mathbf{v})_\alpha| \geqslant |(B\mathbf{w})_\alpha| \qquad \text{for } \alpha \in \{1,2,3,4\} \quad \text{and} \quad |(B\mathbf{v})_k| > |(B\mathbf{w})|_k \tag{4.3}$$

and

$$|(B\mathbf{v}')_\alpha| \geqslant |(B\mathbf{w})_\alpha| \qquad \text{for } \alpha \in \{1,2,3,4\} \backslash \{k\} \quad \text{and} \quad (B\mathbf{v}')_1 > (B\mathbf{w})_1 \tag{4.4}$$

Then $\rho \in \mathcal{G}(\tilde{h})$ for some $\tilde{h} > 0$. On the other hand, suppose that there exists a $\mathbf{w} \in \mathfrak{M}_-$ and even integers N_1, N_2, N_3, N_4, not all zero, such that

$$(B\mathbf{v})^{\mathbf{N}} < (B\mathbf{w})^{\mathbf{N}} \quad \text{for every } \mathbf{v} \in \mathfrak{M}_+ \tag{4.5}$$

where $\mathbf{N} \doteq (N_1, N_2, N_3, N_4)$. Then for every $\tilde{h} \geqslant 0$, $\rho \notin \mathcal{G}(\tilde{h})$.

Proof. To prove the first part of the lemma, we verify the hypotheses of Lemma 1. For each $\mathbf{w} \in \mathfrak{M}_+ \cup \mathfrak{M}_-$, we define

$$c(\mathbf{w}) \doteq \prod_{\alpha=1}^{4} \rho(\{w_\alpha\}) > 0$$

We have for odd $\mathbf{n}$

$$E_\rho\{(B\mathbf{W})^{\mathbf{n}}\} = 2 \sum_{\mathbf{w} \in \mathfrak{M}_+ \cup \mathfrak{M}_-} \delta(\mathbf{w}) c(\mathbf{w}) |(B\mathbf{w})^{\mathbf{n}}| \tag{4.6}$$

where

$$\delta(\mathbf{w}) = \begin{cases} +1 & \text{for } \mathbf{w} \in \mathfrak{M}_+ \\ -1 & \text{for } \mathbf{w} \in \mathfrak{M}_- \end{cases}$$

For each $\mathbf{w} \in \mathfrak{M}_-$, pick vectors in $\mathfrak{M}_+$, $\mathbf{v} = \mathbf{v}(\mathbf{w})$ and $\mathbf{v}' = \mathbf{v}'(\mathbf{w})$, and $k = k(\mathbf{w}) \in \{1,2,3,4\}$ satisfying (4.3)–(4.4). Clearly we may find positive numbers $\{\tilde{c}(\mathbf{v}), \tilde{c}(\mathbf{v}')\}$ such that

$$\sum_{\mathbf{w} \in \mathfrak{M}_+} c(\mathbf{w}) |(B\mathbf{w})^{\mathbf{n}}| \geqslant \sum_{\mathbf{w} \in \mathfrak{M}_-} \left[\tilde{c}(\mathbf{v}) |(B\mathbf{v})^{\mathbf{n}}| + \tilde{c}(\mathbf{v}') |(B\mathbf{v}')^{\mathbf{n}}| \right]$$

[e.g., let $\tilde{c}(\mathbf{w}) = c(\mathbf{w})/(2 \times \text{cardinality of } \mathfrak{M}_-)$ for $\mathbf{w} = \mathbf{v}, \mathbf{v}'$]. We have from (4.6) that

$$E_\rho\{(B\mathbf{W})^{\mathbf{n}}\}/2 \geqslant \sum_{\mathbf{w} \in \mathfrak{M}_-} \left[\tilde{c}(\mathbf{v}) |(B\mathbf{v})^{\mathbf{n}}| + \tilde{c}(\mathbf{v}') |(B\mathbf{v}')^{\mathbf{n}}| - c(\mathbf{w}) |(B\mathbf{w})^{\mathbf{n}}| \right] \tag{4.7}$$

It follows from (4.3) that for each $\mathbf{w} \in \mathfrak{M}_-$, there exists $\hat{n} = \hat{n}(\mathbf{w})$ such that

$$\tilde{c}(\mathbf{v}) |(B\mathbf{v})^{\mathbf{n}}| - c(\mathbf{w}) |(B\mathbf{w})^{\mathbf{n}}| > 0 \qquad \text{if } n_k \geqslant \hat{n}$$

It then follows from (4.4) that there exists $\tilde{n} = \tilde{n}(\mathbf{w})$ such that

$$\tilde{c}(\mathbf{v}') |(B\mathbf{v}')^{\mathbf{n}}| - c(\mathbf{w}) |(B\mathbf{w})^{\mathbf{n}}| > 0 \qquad \text{if } n_k < \hat{n} \quad \text{and} \quad n_1 \geqslant \tilde{n}$$

Consequently we see that the summand in the right-hand side of (4.7) is strictly positive provided only that $n_1 \geqslant \tilde{n}(\mathbf{w})$. Taking $\tilde{n} \doteq \max\{\tilde{n}(\mathbf{w}) \mid \mathbf{w} \in \mathfrak{M}_-\}$, we conclude that the hypotheses of Lemma 1 are valid and thus that $\rho \in \mathcal{G}(\tilde{h})$ for some $\tilde{h} \geqslant 0$.

Concerning the second half of the lemma, we start from the equation

$$\begin{aligned} f(\mathbf{n}; h) &\doteq E_\rho\{(B\mathbf{W})^{\mathbf{n}} \exp[2h(B\mathbf{W})_1]\} \\ &= 2 \sum_{\mathbf{v} \in \mathfrak{M}_+ \cup \mathfrak{M}_-} \delta(\mathbf{v}) c(\mathbf{v}) |(B\mathbf{v})^{\mathbf{n}}| \cosh[2h(B\mathbf{v})_1] \end{aligned} \tag{4.8}$$

which is valid for odd $\mathbf{n}$. We prove that $\rho \notin \mathcal{G}(\tilde{h})$ for any $\tilde{h} \geqslant 0$ by showing that for any $h \geqslant 0$, $f((1,1,1,1)+n\mathbf{N};h) \leqslant 0$ for all large n. Given $h \geqslant 0$, we have from (4.8) that

$$f((1,1,1,1)+n\mathbf{N};h) \leqslant \sum_{\mathbf{v}\in\mathfrak{M}_+} \left[d(\mathbf{v},h)|(B\mathbf{v})^{\mathbf{N}}|^n - \tilde{d}(\mathbf{w},h)|(B\mathbf{w})^{\mathbf{N}}|^n \right] \quad (4.9)$$

where $\mathbf{w}$ (independent of $\mathbf{v}$) is the one satisfying (4.5),

$$d(\mathbf{v},h) \doteq 2c(\mathbf{v})\cosh\left[2h(B\mathbf{v})_1\right] \prod_{\alpha=1}^{4} (B\mathbf{v})_\alpha > 0$$

and

$$\tilde{d}(\mathbf{w},h) \doteq 2c(\mathbf{w})\cosh\left[2h(Bw)_1\right] \left| \prod_{\alpha=1}^{4} (B\mathbf{w})_\alpha \right| / (\text{cardinality of } \mathfrak{M}_+)$$

It now follows from (4.5) that for *any* fixed h, there is some $\tilde{n}(\mathbf{v})$ so that the summand in the right-hand side of (4.9) is strictly negative for $n \geqslant \tilde{n}(\mathbf{v})$. Thus we see that for any h, the left-hand side of (4.9) is strictly negative for all large n [i.e., $n \geqslant \max\{\tilde{n}(\mathbf{v}) \mid \mathbf{v} \in \mathfrak{M}_+\}$]. We conclude that $\rho \notin \mathcal{G}(\tilde{h})$ for any $\tilde{h} \geqslant 0$. ■

Next we state and prove Lemma 5. Given $\mathbf{w} \in \mathbb{R}^4$, we define an ordering $\circledcirc$ among the components $\{w_\alpha\}$ of $\mathbf{w}$ by defining $w_{\alpha_1} \circledcirc w_{\alpha_2}$ if either $w_{\alpha_1} > w_{\alpha_2}$ or $w_{\alpha_1} = w_{\alpha_2}$ and $\alpha_1 < \alpha_2$. For $\alpha \in \{1,2,3,4\}$, we define $w_{(\alpha)}$ to be the αth largest component of $\mathbf{w}$ in accordance with this ordering.

Lemma 5. Let $\mathbf{w} \in \mathbb{R}^4$ satisfy $2(B\mathbf{w})_1 = w_1 + w_2 + w_3 + w_4 > 0$. Then

$$\prod_{\alpha=1}^{4} (B\mathbf{w})_\alpha < 0 \quad (4.10)$$

if and only if $w_{(1)} - w_{(2)} > w_{(3)} - w_{(4)}$. If (4.10) is valid and $\mathbf{v}$ is defined by

$$v_\alpha \doteq \begin{cases} w_\alpha, & \text{if } w_\alpha = w_{(1)} \quad \text{or} \quad w_{(4)} \\ w_\alpha + (w_{(1)} - w_{(2)}), & \text{if } w_\alpha = w_{(2)} \quad \text{or} \quad w_{(3)} \end{cases} \quad (4.11)$$

then

$$\prod_{\alpha=1}^{4} (B\mathbf{v})_\alpha > 0, \qquad (B\mathbf{v})_1 > (B\mathbf{w})_1, \quad \text{and} \quad |(B\mathbf{v})_\alpha| \geqslant |(B\mathbf{w})_\alpha|$$
$$\text{for } \alpha \in \{2,3,4\} \quad (4.12)$$

Proof. We define $\mathbf{u} \in \mathbb{R}^4$ by $u_\alpha \doteq w_{(\alpha)}$, $\alpha \in \{1,2,3,4\}$. Then $\prod_{\alpha=1}^4 (B\mathbf{w})_\alpha = \prod_{\alpha=1}^4 (B\mathbf{u})_\alpha$. We prove the first part of the lemma by showing $\prod_{\alpha=1}^4 (B\mathbf{u})_\alpha < 0$ if and only if $u_1 - u_2 > u_3 - u_4$. If the latter inequality holds, then since $u_1 \geqslant u_2 \geqslant u_3 \geqslant u_4$,

$$\begin{aligned} 2(B\mathbf{u})_1 &= u_1 + u_2 + u_3 + u_4 = w_1 + w_2 + w_3 + w_4 > 0 \\ 2(B\mathbf{u})_2 &= u_1 - u_2 + u_3 - u_4 > 2(u_3 - u_4) \geqslant 0 \\ 2(B\mathbf{u})_3 &= u_1 + u_2 - u_3 - u_4 > 2(u_2 - u_4) \geqslant 0 \\ 2(B\mathbf{u})_4 &= -u_1 + u_2 + u_3 - u_4 < 0 \end{aligned} \tag{4.13}$$

and we see that $\prod_{\alpha=1}^{4}(B\mathbf{u})_\alpha < 0$. If, on the other hand $\prod_{\alpha=1}^{4}(B\mathbf{u})_\alpha < 0$, then $(B\mathbf{u})_\alpha \neq 0$ for $\alpha \in \{1,2,3,4\}$ and since $u_1 \geqslant u_2 \geqslant u_3 \geqslant u_4$, we see that $(B\mathbf{u})_\alpha > 0$ for $\alpha \in \{1,2,3\}$. Thus $(B\mathbf{u})_4$ must be negative, which implies $u_1 - u_2 > u_3 - u_4$. This proves the first part of the lemma.

We now prove the second part of the lemma. Pick $\mathbf{w} \in \mathbb{R}^4$ [with $2(B\mathbf{w})_1 > 0$] satisfying (4.10) and define $\mathbf{v}$ by (4.11). Then for $\alpha \in \{1,2,3,4\}$

$$v_\alpha = \begin{cases} w_{(1)}, & \text{if } w_\alpha = w_{(1)} \quad \text{or} \quad w_{(2)} \\ w_{(1)} - w_{(2)} + w_{(3)}, & \text{if } w_\alpha = w_{(3)} \\ w_{(4)}, & \text{if } w_\alpha = w_{(4)} \end{cases}$$

We first show $\prod_{\alpha=1}^{4}(B\mathbf{v})_\alpha > 0$. Define $\mathbf{x} \in \mathbb{R}^4$ by $x_\alpha \doteq v_{(\alpha)}$, $\alpha \in \{1,2,3,4\}$. Since $\prod_{\alpha=1}^{4}(B\mathbf{v})_\alpha = \prod_{\alpha=1}^{4}(B\mathbf{x})_\alpha$ it suffices to prove $\prod_{\alpha=1}^{4}(B\mathbf{x})_\alpha > 0$. Now $x_1 = x_2 = w_{(1)}$, $x_3 = w_{(1)} - w_{(2)} + w_{(3)}$, and $x_4 = w_{(4)}$. Arguing as in (4.13) and using the fact that, by the first part of the lemma, $w_{(1)} - w_{(2)} > w_{(3)} - w_{(4)}$, one easily finds that $(B\mathbf{x})_\alpha > 0$ for each $\alpha \in \{1,2,3,4\}$. This implies $\prod_{\alpha=1}^{4}(B\mathbf{x})_\alpha > 0$. We now prove that $(B\mathbf{v})_1 \geqslant (B\mathbf{w})_1$ and $|(B\mathbf{v})_\alpha| \geqslant |(B\mathbf{w})_\alpha|$ for $\alpha \in \{2,3,4\}$. Since the $\{w_{(\alpha)}\}$ are a permutation of the $\{w_\alpha\}$, we have

$$2(B\mathbf{v})_1 = v_1 + v_2 + v_3 + v_4 = 3w_{(1)} - w_{(2)} + w_{(3)} + w_{(4)}$$

Using $w_{(1)} - w_{(2)} > w_{(3)} - w_{(4)}$ and $w_{(1)} \geqslant w_{(2)} \geqslant w_{(3)} \geqslant w_{(4)}$, one sees that

$$2(B\mathbf{v})_1 > w_{(1)} + w_{(2)} + w_{(3)} + w_{(4)} = 2(B\mathbf{w})_1$$

The proof that $|(B\mathbf{v})_\alpha| \geqslant |(B\mathbf{w})_\alpha|$ for $\alpha \in \{2,3,4\}$ involves different calculations for the different possible orderings of the $\{w_\alpha\}$. Rather than give the uninteresting details, we consider a representative case. Assuming $w_1 > w_2 > w_3 > w_4$, one finds $(B\mathbf{v})_2 = (B\mathbf{w})_2 > 0$, $(B\mathbf{v})_3 = (B\mathbf{w})_3 > 0$,

$$2(B\mathbf{v})_4 = w_1 - w_2 + w_3 - w_4 \geqslant w_1 - w_2 - w_3 + w_4 = 2|(B\mathbf{w})_4| \qquad \blacksquare$$

Proof of Theorem 3. The case $r = 2$ is covered by Ref. 3, Theorem 1.2(a), and the case $r = 3$ is covered by Ref. 3, Theorem 1.2(b) and by Theorem 4, which we prove below. We prove the cases $r = 4$ and $r = 5$ by using the first part of Lemma 4; a somewhat different proof of these cases is given in Ref. 8, Chapter 4. In the case $r = 4$, we may assume without loss of generality that [in the notation of (2.5)] $m_1 = -1$, $m_2 = -1 + \eta$, $m_3 = 1 - \eta$, $m_4 = 1$ for some $\eta \in (0, 1)$. There are four basic possibilities for $\mathbf{w} \in \mathfrak{M}_-$ if $\eta \in (2/3, 1)$ and two basic possibilities if $\eta \in (0, 2/3]$ (together with others obtained by permutation of the components). These possibilities may be easily listed by using the first part of Lemma 5. For each of these basic $\mathbf{w}$'s, we will display the corresponding $\mathbf{v}$ and $\mathbf{v}'$ satisfying (4.3)–(4.4); permutations of the basic $\mathbf{w}$'s are handled by correspondingly permuting the $\mathbf{v}$ and $\mathbf{v}'$:

for $\mathbf{w} = (1, 1 - \eta, 1 - \eta, 1 - \eta)$, $\quad \mathbf{v} = \mathbf{v}' = (1, 1, 1, 1 - \eta)$

for $\eta > 2/3$ and $\mathbf{w} = (1, 1 - \eta, 1 - \eta, -1 + \eta)$, $\quad \mathbf{v} = \mathbf{v}' = (1, 1, 1, -1)$

for $\mathbf{w} = (1, 1 - \eta, -1 + \eta, -1 + \eta)$,

$\quad \mathbf{v} = (1, 1, -1, -1 + \eta)$ and $\mathbf{v}' = (1, 1, 1, -1)$

for $\eta > 2/3$ and $\mathbf{w} = (1, -1 + \eta, -1 + \eta, -1 + \eta)$,

$\quad \mathbf{v} = \mathbf{v}' = (1, 1, 1, -1)$

In the case $r = 5$, we may take $m_1 = -1$, $m_2 = -1 + \eta$, $m_3 = 0$, $m_4 = 1 - \eta$, $m_5 = 1$. For $\mathbf{w}$'s in $\mathfrak{M}_-$ with no vanishing components, we may take $\mathbf{v}$, $\mathbf{v}'$ as described above for $r = 4$. There are seven basic possibilities for the remaining $\mathbf{w}$'s in $\mathfrak{M}_-$ if $\eta \in (1/2, 1)$ and four basic possibilities if $\eta \in (0, 1/2]$:

$$(1, 1 - \eta, 0, 0), \qquad (1, 0, 0, 0), \qquad (1, 0, 0, -1 + \eta), \qquad (1 - \eta, 0, 0, 0)$$
$$(1, 1 - \eta, 1 - \eta, 0), \qquad (1, 1 - \eta, 0, -1 + \eta), \qquad (1, 0, -1 + \eta, -1 + \eta)$$

where the latter three are valid only for $\eta > 1/2$. It turns out that for each of these seven possibilities we may take $\mathbf{v} = \mathbf{v}' = (1, 1, 1, -1)$.

To prove part (c) of Theorem 3, we note that by the first part of Lemma 4 (with $\mathbf{v} = \mathbf{v}'$ and $k = 1$) together with Lemma 5, it suffices if for each w_1, w_2, w_3, w_4 in the support of ρ, $w_{(3)} + (w_{(1)} - w_{(2)}) = w_{(1)} - (w_{(2)} - w_{(3)})$ is also in the support of ρ [cf. (4.11) with $\alpha = 3$]. But this is clearly the case when the support points of ρ are equally spaced.

It remains to construct the $\bar{\rho}$ of part (d) of the theorem. We will take $m_6 = 1/2$, $m_5 = 1/2 - \epsilon^2$, and $m_4 = 1/2 - \epsilon$ and show that for suitably chosen small positive ϵ, the second half of Lemma 4 can be applied with

$$\mathbf{w} = (m_6, m_4, m_5, m_4) \quad \text{and} \quad \mathbf{N} = (2[\kappa|\ln \epsilon|], 2, 0, 0) \tag{4.14}$$

where the notation $[-]$ denotes the usual greatest integer function and $\kappa > 0$ is to be determined later. With $\mathbf{w}$ as in (4.14), we have $\mathbf{w} \in \mathfrak{M}_-$ (for small ϵ) by Lemma 5 and

$$(B\mathbf{w})_1 = 1 - \epsilon - \epsilon^2/2, \qquad (B\mathbf{w})_2 = \epsilon - \epsilon^2/2 \tag{4.15}$$

On the other hand, one finds (by listing possibilities) that for every $\mathbf{v} \in \mathfrak{M}_+$, either

$$(B\mathbf{v})_1 = 1 - \epsilon/2 + O(\epsilon^2), \qquad |(B\mathbf{v})_2| = \epsilon/2 + O(\epsilon^2) \tag{4.16}$$

[for $\mathbf{v} = (m_6, m_6, m_6, m_4)$ or (m_6, m_6, m_5, m_4) or (m_6, m_5, m_5, m_4) or (m_5, m_5, m_5, m_4) or permutations of these] or else

$$(B\mathbf{v})_1 = 1 + O(\epsilon^2), \qquad |(B\mathbf{v})_2| = O(\epsilon^2) \tag{4.17}$$

[for $\mathbf{v} = (m_6, m_6, m_6, m_5)$ or its permutations] or else

$$(B\mathbf{v})_1 \leqslant \tfrac{1}{2} + O(\epsilon), \qquad |(B\mathbf{v})_2| \leqslant 1 \tag{4.18}$$

(for all other $\mathbf{v}$'s in $\mathfrak{M}_+$). In order to conclude that for sufficiently small ϵ, (4.5) will be valid with $\mathbf{w}$, $\mathbf{N}$ given by (4.14), it suffices to show that for every $\mathbf{v} \in \mathfrak{M}_+$,

$$\overline{K}_\mathbf{v} \doteq \limsup_{\epsilon\to 0} (B\mathbf{v})^\mathbf{N}/\epsilon^2 < \lim_{\epsilon\to 0} (B\mathbf{w})^\mathbf{N}/\epsilon^2 \doteq \overline{K}_\mathbf{w} \tag{4.19}$$

But using the definition of $\mathbf{N}$, we have from (4.15) that

$$\overline{K}_\mathbf{w} = \lim_{\epsilon\to 0} \left\{ \left[1 - \epsilon + O(\epsilon^2) \right]^{2\kappa|\ln \epsilon| + O(1)} \cdot \left[\epsilon + O(\epsilon^2) \right]^2 / \epsilon^2 \right\} = 1 \tag{4.20}$$

while we have similarly that

$$\overline{K}_\mathbf{v} = \begin{cases} 1/4, & \text{if (4.16) is valid} \\ 0, & \text{if (4.17) is valid} \end{cases}$$

If (4.18) is valid, then

$$\overline{K}_\mathbf{v} \leqslant \text{const} \times \left[\lim_{\epsilon\to 0} (1/2)^{2\kappa|\ln \epsilon|}/\epsilon^2 \right]$$

Thus if we choose κ so that $2\kappa \ln 2 > 2$, we see that $\overline{K}_\mathbf{v} = 0$ if (4.18) is valid. This yields (4.19) and completes the proof of Theorem 3. ■

Proof of Theorem 4. A direct calculation [cf. (4.8)] gives for n_2, n_3, n_4 all odd that

$$f(\mathbf{n}; h) = 2^{-(n_1+n_2+n_3+n_4)} \{ c_1 [g(3h) 3^{n_1} + g(h)(3^{n_2} + 3^{n_3} + 3^{n_4})] + c_2 g(2h) 2^{n_1+n_2+n_3+n_4} - c_3 g(h) \} \tag{4.21}$$

where

$$g(h) \doteq g_{n_1}(h) \doteq \begin{cases} 2\cosh h, & \text{if } n_1 \text{ is odd} \\ 2\sinh h, & \text{if } n_1 \text{ is even} \end{cases}$$

and

$$c_1 \doteq (1-a)^3 a/2, \qquad c_2 \doteq (1-a)^4/4, \qquad c_3 \doteq 2(1-a)a^3$$

Since $f(\mathbf{n}; h)$ vanishes unless n_2, n_3, n_4 are all odd, in order that $f(\mathbf{n}; h) \geqslant 0$ for all $\mathbf{n}$ it suffices if

$$c_1\left[g(3h)3^{n_1} + g(h)(3^{n_2} + 3^{n_3} + 3^{n_4})\right] + c_2 g(2h)2^{n_1+n_2+n_3+n_4} \geqslant c_3 g(h) \tag{4.22}$$

for all n_1 and odd n_2, n_3, n_4. But for $h > 0$ and n_1 restricted to be even (respectively, odd), the left-hand side of (4.22) is increasing in the n_i's, and so it suffices to check the cases $n_2 = n_3 = n_4 = 1$ and $n_1 = 0$ or 1:

$$c_1 3^{n_1} g(3h)/g(h) + 8c_2 2^{n_1} g(2h)/g(h) + 9c_1 - c_3 \geqslant 0 \tag{4.23}$$

It is an elementary fact that for $h > 0$

$$k \cosh(kh)/\cosh(h) \geqslant \sinh(kh)/\sinh(h), \qquad k \in \{1, 2, 3, \ldots\}$$

(To prove this, cross-multiply and compare Taylor expansions about $h = 0$.) Hence it suffices to check (4.23) for $n_1 = 0$ or equivalently,

$$u^2 + 2Au + (2 - 1/A^2) \geqslant 0 \tag{4.24}$$

where

$$u \doteq \cosh h, \qquad A \doteq 2c_2/c_1 = (1 - a)/a$$

Since the roots of the polynomial in (4.24) are $-1/A$ and $(1/A) - 2A$ we see that for $A \geqslant 1/2$ (i.e., $a \leqslant 2/3$), (4.24) is valid for any $u \geqslant 1$ (i.e., any $h \geqslant 0$). Hence $\rho_a \in \mathcal{G}(0)$ for $a < 2/3$. On the other hand, for $A < 1/2$ (i.e., $a > 2/3$), (4.24) is valid provided

$$u \doteq \cosh h \geqslant (1/A) - 2A = 1 + \frac{3a - 2}{a(1 - a)} \tag{4.25}$$

We may thus take $\tilde{h}$ as in (2.7) and conclude that $\rho_a \in \mathcal{G}(\tilde{h})$. ■

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